# A HELLY-TYPE PROBLEM

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Abstract. We raise a question related to Helly's theorem with the added elements of geometric transformations.

The classic theorem of Helly tells us the following fact: if any $d+1$ among $n$ convex bodies in $\mathbb{R}^d$ have non-empty intersection, then all $n$ of them have non empty-intersection. $d+1$ is the best constant for a general family of $d$-dimensional convex body, and so in a way the theorem gives another interpretation for the concept of dimensions. This simple looking result has served to inspire many great developments in discrete geometry.

A general format that applies to all Helly-related theorems is as follows: for a collection geometric objects with sufficiently generic properties, if any $k$-subcollection satisfies some special property, then so does the whole collection. Usually, such a minimal threshold $k$ exists and depends on the particular class of objects. For convex bodies, it is the number of dimensions plus 1. For non–convex bodies with sufficiently nice intersections, there is still a corresponding threshold, although not solely dependent upon the dimension. The "special property" mentioned can vary from non-empty intersection to common transversal line, etc. For a good reference, please look at [Mat] or [VGG].

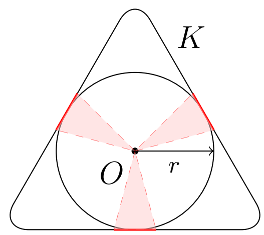
We look at another situation where certain transformations are allowed to act on the objects, in this paper it is rotations. First, let us state a straightforward result following from Helly's theorem:

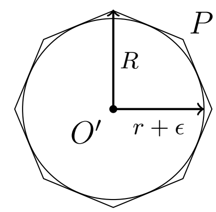
**Corollary 1.** *Given $n$ points in $\mathbb{R}^d$, if any $d+1$ among them can be covered by a certain translate of a convex body $K$ then all $n$ points can also be covered in that way.*

*Proof.* A translate $t+K$ contains a point $v$ if and only if $t \in v+(-K)$. Forming a new family $\{v_i+(-K)\}$, the given condition is now equivalent to any $d+1$ family members having non-empty intersection. Helly's theorem guarantees a point belonging to all the objects and this is the translation vector that makes $K$ cover each $v_i$. $\square$

In this corollary's setting, what if we allow both translations and rotations of the body $K$, is there still a similar result? This of course does not say anything new if $K$ is a round ball, but if $K$ is any other convex body, is there still another $k$ in place of $d+1$. Should we expect that such statement is true then $k$ must really depends on the particular shape of $K$ and can be very large for special bodies. Ironically, we are going to show that such a result would never exist, at least for a large family of convex sets. Call the largest circle inscribable in $K$ its incircle, we have:

**Lemma 2.** *Let $O$ be $K$'s incircle (possibly one among many), if $\partial K \cup O$ is a discrete set then no such number $k$ exists.*




*Proof.* In the above two figures, the left one shows a two dimensional convex body $K$ with the incircle $O$ having radius $r$. On the right we have a slightly larger circle $O'$ with radius $r + \epsilon$, also there is a regular polygon $P$ containing $O'$ as its incircle. $P$ has $n$ edges touching $O'$ and the distance from its vertices to the center of $O'$ is $R$. We have $\lim_{\substack{\epsilon \to 0 \\ n \to \infty}} R = r$ but clearly any such $P$ cannot be contained in $K$. Assume $n$ and $\epsilon$ are respectively big and small enough, we mark on the boundary of $K$ any point that has distance less than $R$ away from the center of $O$. Let $\alpha$ be the total measure of the angles subtended radically by these marked points, then $\lim_{\substack{\epsilon \to 0 \\ n \to \infty}} \alpha = 0$. We first translate the two figures so that $O$ and $O'$ are concentric. Now for any vertex of $P$, we can properly rotate $P$ to make sure that it avoids all marked portions and thus lies in side $K$. The possible set of rotations for this vertex measures $2\pi - \alpha$. Hence, we choose any simultaneously rotate any $k$ among $n$ vertices in this way provided that $k\alpha < 2\pi$. As $\alpha$ approaches 0, $k$ can be as large as possible but still the whole polygon $K$ is not inscribable in $K$ after any rotation. $\square$

The above proof is also adaptable to all higher dimensional convex bodies whose intersections with boundaries of inner circles have zero measure. This rules out all convex polytopes as well as simple cases like a sphere with pieces sliced away by hyperplanes. We are left with cases when $\partial K$ shares a positive-measured intersection with its incircle. So far the author knows of no effective method to construct counterexamples for these remaining cases. Thinking from the opposite perspective, if we denote $\alpha$ as the $(d-1)$-dimensional measure of $\partial K \cap O$ and $\beta$ as that of $O$, then such a $k$ if exists should be bounded below by $\frac{\beta}{\alpha}$.

## References

[VGG] E. C. de Verdière, G. Ginot, X. Goaoc. Helly numbers of acyclic families. *ArXiv:1101.6006*, [math.CO], 2011.

[Mat] J. Matoušek. Lectures on Discrete Geometry. *Springer-Verlag*, New York, 2002.